

# FACES OF WEIGHT POLYTOPES AND A GENERALIZATION OF A THEOREM OF VINBERG

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**ABSTRACT.** The paper is motivated by the study of graded representations of Takiff algebras, cominuscule parabolics, and their generalizations. We study certain special subsets of the set of weights (and of their convex hull) of the generalized Verma modules (or GVM's) of a semisimple Lie algebra  $\mathfrak{g}$ . In particular, we extend a result of Vinberg and classify the faces of the convex hull of the weights of a GVM. When the GVM is finite-dimensional, we answer a natural question that arises out of Vinberg's result: when are two faces the same?

We also extend the notion of interiors and faces to an arbitrary subfield  $\mathbb{F}$  of the real numbers, and introduce the idea of a weak  $\mathbb{F}$ -face of any subset of Euclidean space. We classify the weak  $\mathbb{F}$ -faces of all lattice polytopes, as well as of the set of lattice points in them. We show that a weak  $\mathbb{F}$ -face of the weights of a finite-dimensional  $\mathfrak{g}$ -module is precisely the set of weights lying on a face of the convex hull.

## 1. INTRODUCTION

In this note, we study the faces of the convex hull of the weights of a highest weight representation  $V$  of a complex semisimple Lie algebra  $\mathfrak{g}$ . The classification of the faces in the case when  $V$  is a simple finite-dimensional representation of  $\mathfrak{g}$  had been obtained by Vinberg [Vin]. Roughly speaking, his result states that a face of the weight polytope of a simple finite-dimensional representation is determined by a pair consisting of an element of the Weyl group and a subset of the set of simple roots. Our results extend (and recover) those of Vinberg's for arbitrary generalized Verma modules. Our methods, however, are completely different and rely on algebra and convexity theory. In particular, we are able to work with convex linear combinations of the weights, where the coefficients are in an arbitrary subfield of the real numbers. We are also able to answer a natural question arising from Vinberg's result: namely, when do two different pairs give rise to the same face of the weight polytope of a finite-dimensional simple Lie algebra.

This paper was motivated by the results in [CG] (which are further extended in [CKR]) on representations of Takiff algebras and their generalizations. In those papers, one showed that one could associate Koszul algebras in a natural fashion, to certain subsets of the set of weights of a finite-dimensional representation of a semisimple Lie algebra. In this paper, we show that the conditions on these subsets is exactly equivalent to requiring the subset to be the maximal subset of weights contained in a face. This description generalizes and makes uniform the results of [CDR], where the case of the adjoint representation was analyzed.

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*Date:* December 14, 2010.

*2000 Mathematics Subject Classification.* Primary: 17B20; Secondary: 17B10.

*Key words and phrases.* Weak  $\mathbb{F}$ -face, positive weak  $\mathbb{F}$ -face, generalized Verma module, polyhedron.

**Organization.** The paper is organized as follows. In Section 2, we study generalized Verma modules. These are a family of highest weight  $\mathfrak{g}$ -modules, that run from all Verma modules at one end, to all finite-dimensional simple modules at the other. The convex hull of their set of weights turns out always to be a polyhedron, and our main goal in this section is to classify their faces, in terms of describing the vertices and the extremal rays. This generalizes Vinberg's result from [Vin].

For the rest of the paper, we focus on finite-dimensional  $\mathfrak{g}$ -modules  $V$ . We wish to study the subsets of weights of  $V$ , which lie on faces of the convex hull of all weights. To that end, we introduce the notion of a *weak face*, over an arbitrary subfield  $\mathbb{F} \subset \mathbb{R}$ . Among these weak  $\mathbb{F}$ -faces, we then consider *positive* weak  $\mathbb{F}$ -faces. In Section 3, we classify the (positive) weak  $\mathbb{F}$ -faces of  $V$ . This generalizes results from [CDR, CG], which addressed the example of  $V = \mathfrak{g}$ .

In Section 4, we study (positive) weak  $\mathbb{F}$ -faces of arbitrary subsets  $X \subset \mathbb{R}^n$ . Our main results here concern the case when the convex hull of  $X$  is a polyhedron. In this case, the (positive) weak  $\mathbb{F}$ -faces are precisely the elements of  $X$  that lie on a proper face of the polyhedron - in other words, that maximize a linear functional, with finite (positive) maximum.

Finally, in Section 5, we prove our results from Section 3, using the techniques developed in Section 4.

**Acknowledgements.** The authors are especially grateful to Vyjayanthi Chari for extremely valuable discussions and her many inputs and suggestions, that helped bring this manuscript to its present form. The first author would also like to thank Michel Brion, Shrawan Kumar, and Olivier Mathieu for valuable discussions.

## 2. RESULTS ON GENERALIZED VERMA MODULES

Throughout this paper, we let  $\mathbb{R}$  (respectively  $\mathbb{Q}$ ,  $\mathbb{Z}$ ) denote the real numbers (respectively the rationals, and the integers). For any subset  $R \subset \mathbb{R}$ , we let  $R_+ := R \cap [0, \infty)$ ,  $R_{>0} := R \cap (0, \infty)$ . If  $A, B \subset V$  are subsets of an abelian group  $(V, +)$ , we define their Minkowski sum to be  $A + B := \{a + b : a \in A, b \in B\} \subset V$ . (If  $A = \{a\}$ , we may also write this as  $a + B$ .) Similarly,  $-B := \{-b : b \in B\}$ .

**2.1.** Fix a complex semisimple Lie algebra  $\mathfrak{g}$  of rank  $n$  and a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ , and let  $\Phi \subset \mathfrak{h}^*$  be the set of roots of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ . Set  $I = \{1, \dots, n\}$  and fix a set  $\{\alpha_i : i \in I\}$  of simple roots. Denote by  $\Phi^+$  the corresponding set of positive roots. Let  $\kappa$  be the Killing form on  $\mathfrak{g}$ ; recall that its restriction to  $\mathfrak{h}$  induces a positive definite inner product  $(\cdot, \cdot)$  on the real span  $\mathfrak{h}_{\mathbb{R}}^*$  of  $\Phi^+$ . Let  $\{\omega_i : i \in I\}$  be the basis of  $\mathfrak{h}^*$  which satisfies  $2(\alpha_i, \omega_j) = \delta_{i,j}(\alpha_i, \alpha_i)$ . Since the Killing form is nondegenerate, it induces an identification of  $\mathfrak{h}_{\mathbb{R}}$  with  $\mathfrak{h}_{\mathbb{R}}^*$ . Define  $h_{\alpha_i} \in \mathfrak{h}_{\mathbb{R}}$  to be the vector identified with  $2\alpha_i/(\alpha_i, \alpha_i)$ ; these vectors form an  $\mathbb{R}$ -basis of  $\mathfrak{h}_{\mathbb{R}}$ .

The root lattice  $Q$  (respectively, weight lattice  $P$ ) is the integer span of the simple roots  $\alpha_i$  (respectively, fundamental weights  $\omega_i$ ), while  $Q^+$  (respectively,  $P^+$ ) is the  $\mathbb{Z}_+$ -span of the simple roots (respectively, fundamental weights). Given a subset  $J$  of  $I$ , let  $Q_J$  (respectively  $P_J$ ) be the  $\mathbb{Z}$ -span of the simple roots  $\{\alpha_j : j \in J\}$  (respectively, the fundamental weights  $\{\omega_j : j \in J\}$ ), and set  $\Phi_J^+ := \Phi^+ \cap Q_J$ ,  $P_J^+ := P^+ \cap P_J$ ,  $Q_J^+ := Q^+ \cap Q_J$ .

Given any  $\lambda \in \mathfrak{h}^*$ , say  $\lambda = \sum_{i \in I} r_i \omega_i$  with all  $r_i \in \mathbb{R}$ , we set

$$\text{supp } \lambda := \{i \in I : r_i \neq 0\}, \quad J_\lambda := \{i \in I : \lambda(h_{\alpha_i}) \in \mathbb{Z}_+\}.$$

Clearly,  $\lambda \in P^+$  if and only if  $J_\lambda = I$ . Finally, let  $W$  be the Weyl group of  $\Phi$ , namely the subgroup of  $\text{Aut}(\mathfrak{h}_\mathbb{R}^*)$  generated by the simple reflections  $\{s_i : i \in I\}$ . Note that the inner product  $(\cdot, \cdot)$  on  $\mathfrak{h}_\mathbb{R}^*$  is  $W$ -invariant.

**2.2.** Fix a Chevalley basis  $\{x_\alpha^\pm, h_i = h_{\alpha_i} : \alpha \in \Phi^+, 1 \leq i \leq n\}$  of  $\mathfrak{g}$ , set  $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Phi^\pm} \mathbb{C}x_\alpha^\pm$ , and write

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

The subalgebras  $\mathfrak{n}_J^\pm$  are defined in the obvious way. Let  $\mathfrak{p}_J$  be the parabolic Lie subalgebra of  $\mathfrak{g}$ , defined as follows:

$$\mathfrak{p}_J = \mathfrak{n}_J^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{m}_J = \mathfrak{n}_J^- \oplus \mathfrak{h} \oplus \mathfrak{n}_J^+, \quad \mathfrak{u}_J^\pm = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi_J^+} \mathbb{C}x_\alpha^\pm,$$

where  $\mathfrak{m}_J$  is reductive with semisimple part  $\mathfrak{g}_J$ , and  $\mathfrak{u}_J^\pm$  is nilpotent. The subgroup  $W_J$  of  $W$  generated by  $\{s_j : j \in J\}$  is the Weyl group of  $\mathfrak{g}_J$ , and we set  $\rho_J = \sum_{j \in J} \omega_j$ . The following is standard, but we isolate it in the form of a Lemma, since it is used frequently in the paper.

**Lemma.** For  $w \in W_J$  and  $i \notin J$ , we have  $w\alpha_i \in \Phi^+$ , and hence  $w\alpha \in \Phi^+$  for all  $\alpha \in \Phi^+ \setminus \Phi_J^+$ .

Given any Lie algebra  $\mathfrak{a}$ , we let  $\mathbf{U}(\mathfrak{a})$  be the universal enveloping algebra of  $\mathfrak{a}$ . The Poincaré–Birkhoff–Witt theorem gives us an isomorphism of vector spaces:

$$\mathbf{U}(\mathfrak{g}) \cong \mathbf{U}(\mathfrak{n}^-) \otimes \mathbf{U}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{n}^+).$$

**2.3.** We now recall the definition and elementary properties of the *generalized Verma modules*. Recall that a weight module  $V$  for a reductive Lie algebra  $\mathfrak{a}$  with Cartan subalgebra  $\mathfrak{t}$  is one which has a decomposition

$$V = \bigoplus_{\mu \in \mathfrak{t}^*} V_\mu,$$

where  $V_\mu = \{v \in V : hv = \mu(h)v, \forall h \in \mathfrak{t}\}$ . We set  $\text{wt } V = \{\mu \in \mathfrak{t}^* : V_\mu \neq 0\}$ .

Given  $\lambda \in \mathfrak{h}^*$  and  $J \subset J_\lambda$ , the generalized Verma module  $M(\lambda, J)$  is the  $\mathfrak{g}$ -module generated by an element  $m_\lambda$  with defining relations:

$$\mathfrak{n}^+ m_\lambda = 0, \quad h m_\lambda = \lambda(h) m_\lambda, \quad (x_\alpha^-)^{\lambda(h_\alpha)+1} m_\lambda = 0,$$

for all  $h \in \mathfrak{h}$  and  $\alpha \in \Phi_J^+$ . The following is standard - see [Kum]:

**Proposition.** Let  $\lambda \in \mathfrak{h}^*$  and  $J \subset J_\lambda$ .

- (i) The  $\mathfrak{g}$ -module  $M(\lambda, J)$  is a free  $\mathbf{U}(\mathfrak{u}_J^-)$ -module, and  $\dim \mathbf{U}(\mathfrak{m}_J) m_\lambda < \infty$ . In particular,  $\text{wt}(\mathbf{U}(\mathfrak{m}_J) m_\lambda)$  is a finite subset of  $\mathfrak{h}^*$ , and

$$\text{wt } M(\lambda, J) = \text{wt}(\mathbf{U}(\mathfrak{m}_J) m_\lambda) - \left\{ \sum_{\alpha \notin \Phi_J^+} r_\alpha \alpha : r_\alpha \in \mathbb{Z}_+ \right\}.$$

(ii) *The set  $\text{wt } M(\lambda, J)$  is  $W_J$ -invariant.*

In the special case when  $\lambda \in P^+$ , the module  $M(\lambda, I)$  is the irreducible finite-dimensional module with highest weight  $\lambda$ .

**2.4.** Given any subset  $X$  of  $\mathfrak{h}_{\mathbb{R}}^*$  we let  $\text{conv}_{\mathbb{R}}(X)$  be the convex hull of  $X$ ; i.e.,

$$\text{conv}_{\mathbb{R}}(X) = \left\{ \sum_{s=1}^k r_s x_s : k \in \mathbb{Z}_+, r_s \in \mathbb{R}_+, x_s \in X, \sum_{s=1}^k r_s = 1 \right\}.$$

Also define  $\text{cone}_{\mathbb{R}}(X)$  to be the cone of  $X$ , i.e.,

$$\text{cone}_{\mathbb{R}}(X) = \left\{ \sum_{s=1}^k r_s x_s : k \in \mathbb{Z}_+, r_s \in \mathbb{R}_+, x_s \in X \right\}.$$

**Proposition.** *For  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$  and  $J \subset J_{\lambda}$ , we have*

$$\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J)) = \text{conv}_{\mathbb{R}}(\text{wt } \mathbf{U}(\mathfrak{m}_J)m_{\lambda}) - \text{cone}_{\mathbb{R}}(\Phi^+ \setminus \Phi_J^+),$$

*and hence  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$  is a  $W_J$ -invariant subset of  $\mathfrak{h}_{\mathbb{R}}^*$ .*

*Proof.* It is clear (by Proposition 2.3) that  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$  is contained in the right hand side. For the reverse inclusion, let

$$\mu = \sum_k r_k \mu_k - \sum_{\alpha \in \Phi^+ \setminus \Phi_J^+} m_{\alpha} \alpha, \quad \mu_k \in \text{wt } \mathbf{U}(\mathfrak{m}_J)m_{\lambda}, \quad r_k, m_{\alpha} \in \mathbb{R}_+, \quad \sum_k r_k = 1.$$

If  $m_{\alpha} = 0$  for all  $\alpha$ , then we are done since  $\mu_k \in \text{wt}(M(\lambda, J))$  for all  $k$ . Hence, we may assume that  $m_{\alpha} > 0$  for some  $\alpha \in \Phi^+ \setminus \Phi_J^+$ . Furthermore, we may assume without loss of generality that  $r_1 \neq 0$ . Thus, we can write

$$\mu = \sum_{k \neq 1} r_k \mu_k + r_1 \left( \mu_1 - \sum_{\alpha} \frac{m_{\alpha}}{r_1} \alpha \right).$$

Choose  $t \in \mathbb{Z}_+$  such that  $r = \sum_{\alpha} \frac{m_{\alpha}}{r_1} \leq t$ . Since  $\mu_1 - t\alpha \in \text{wt } M(\lambda, J)$ , the claim follows by noting that

$$\mu = \sum_{k \neq 1} r_k \mu_k + \left( r_1 - \frac{r_1 r}{t} \right) \mu_1 + \sum_{\alpha} \frac{m_{\alpha}}{t} (\mu_1 - t\alpha) \in \text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J)).$$

The fact that  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$  is  $W_J$ -invariant is immediate from Proposition 2.3.  $\square$

**2.5.** We now need some more notions from convexity theory. Given  $v, w \in \mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^n$ , define the (affine) hyperplane and the corresponding half-space as follows:

$$H(v, w) := \{u \in \mathbb{R}^n \mid v \cdot (u - w) = 0\}, \quad H^+(v, w) := \{u \in \mathbb{R}^n \mid v \cdot (u - w) \geq 0\}.$$

Let  $\mathcal{P}$  be a (nonempty) subset of  $\mathbb{R}^n$ . We say that  $H(v, w)$  is a *supporting hyperplane* of  $\mathcal{P}$  if

$$\mathcal{P} \subset H^+(v, w) \text{ and } \mathcal{P} \cap H(v, w) \neq \emptyset.$$

A *face* of  $\mathcal{P}$  is  $\mathcal{P}$  or the intersection of  $\mathcal{P}$  with a supporting hyperplane. We will say that  $\mathcal{P}$  is a *polyhedron* if it is the intersection of a finite number of affine half-spaces, and a bounded polyhedron is a *polytope*. The following is standard; see [Zie], for instance:

**Theorem** (Decomposition Theorem). *Let  $\mathcal{P}$  be a subset of  $\mathbb{R}^n$ . Then,*

- (i) (Weyl-Minkowski Theorem.)  *$\mathcal{P}$  is a polytope if and only if  $\mathcal{P} = \text{conv}_{\mathbb{R}}(U)$  for some finite subset  $U \subset \mathbb{R}^n$ .*
- (ii) (Finite Basis Theorem.)  *$\mathcal{P}$  is a polyhedron if and only if  $\mathcal{P} = \text{conv}_{\mathbb{R}}(U) + \text{cone}_{\mathbb{R}}(V)$  for some finite sets  $U, V \subset \mathbb{R}^n$ .*

In particular, the convex hull of the union of a finite set with a polytope is also a polytope.

Using the Decomposition Theorem, we have the following corollary to Proposition 2.4:

**Corollary.** *The set  $\text{conv}_{\mathbb{R}}(\text{wt } \mathbf{U}(\mathfrak{m}_J)m_\lambda)$  is a convex polytope, and  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$  is a convex polyhedron in  $\mathfrak{h}_{\mathbb{R}}^*$ .*

**2.6.** One of the main results of this paper is the following:

**Theorem 1.** *Let  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ ,  $J \subset J_\lambda$ , and let  $F \subset \text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$ . Then  $F$  is a face of  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$  if and only if there exists a subset  $I_0$  of  $I$  and  $w \in W_J$ , such that*

$$wF = \text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J) \cap (\lambda - Q_{I_0}^+)).$$

*Proof.* Let  $F$  be a face of  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$ . By Lemma 4.2,  $F$  maximizes some linear functional  $\varphi \in (\mathbb{R}^n)^*$  in  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$ . Let  $\nu \in \mathfrak{h}_{\mathbb{R}}^*$  be such that  $\varphi(\mu) = (\nu, \mu)$  for all  $\mu \in \mathfrak{h}_{\mathbb{R}}^*$ . Choose  $w \in W_J$  such that  $(w(\nu), \alpha_j) \geq 0$  for all  $j \in J$ . Notice that  $wF$  maximizes the inner product  $(w(\nu), -)$  and, hence, is a face of  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$ .

Suppose that  $(w(\nu), \alpha_i) < 0$  for some  $i \in I \setminus J$ . Since  $i \notin J$ ,  $\lambda - r\alpha_i \in \text{wt } M(\lambda, J)$  for all  $r \in \mathbb{Z}_+$ . However,  $(w(\nu), \lambda - r\alpha_i) = (w(\nu), \lambda) - r(w(\nu), \alpha_i)$  can be arbitrarily large, which contradicts  $(w(\nu), -)$  having a finite maximum in  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$ . Thus,  $w(\nu)$  must be in the fundamental Weyl chamber.

Write  $w(\nu) = \sum_{i \in I} a_i \omega_i$  with  $a_i \geq 0$  for all  $i \in I$ . Letting  $I_0 = \{i \in I \mid a_i = 0\}$ , it is clear that  $wF = \text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J) \cap (\lambda - Q_{I_0}^+))$ .

For the converse, let  $\rho_{I \setminus I_0} = \sum_{i \in I \setminus I_0} \omega_i$ , and consider the linear functional  $\varphi$  given by

$$\varphi(\mu) = (\rho_{I \setminus I_0}, \mu).$$

The subset of  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$  that maximizes  $\varphi$  is precisely  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J) \cap (\lambda - Q_{I_0}^+))$ . Hence  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J) \cap (\lambda - Q_{I_0}^+))$  is a face of  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$ .

Suppose that  $F \subset \text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$  and  $wF = \text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J) \cap (\lambda - Q_{I_0}^+))$  for some  $w \in W_J$ . Notice that  $F$  maximizes the linear functional  $\varphi \circ w$  where  $\varphi(\mu) = (\rho_{I \setminus I_0}, \mu)$  as above; therefore,  $F$  is a face of  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$  by Lemma 4.2.  $\square$

As a consequence, we obtain information about the set of weights of  $M(\lambda, J)$  that lie in a face.

**Corollary.** *Let  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ ,  $J \subset J_\lambda$ , and suppose that  $F$  is a face of  $\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J))$ . There exist  $w \in W_J$  and  $I_0 \subset I$  such that*

$$w(F \cap \text{wt } M(\lambda, J)) = \text{wt } M(\lambda, J) \cap (\lambda - Q_{I_0}^+).$$

*Proof.* By Theorem 1, there exist  $w \in W_J$  and  $I_0 \subset I$ , such that

$$wF = \text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J) \cap (\lambda - Q_{I_0}^+)).$$

By Proposition 2.3,  $\text{wt } M(\lambda, J)$  is  $W_J$ -invariant, and so we have

$$wF \cap \text{wt } M(\lambda, J) = w(F \cap \text{wt } M(\lambda, J)),$$

and the corollary follows if we prove that

$$\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J) \cap (\lambda - Q_{I_0}^+)) \cap \text{wt } M(\lambda, J) = (\lambda - Q_{I_0}^+) \cap \text{wt } M(\lambda, J).$$

It is clear that the right hand side is contained in the left hand side. For the reverse inclusion, given  $\mu \in \text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J) \cap (\lambda - Q_{I_0}^+)) \cap \text{wt } M(\lambda, J)$ , we write:

$$\mu = \lambda - \sum_{i \in I} n_i \alpha_i = \sum_{s=1}^k a_s (\lambda - \sum_{i \in I_0} m_{si} \alpha_i),$$

where  $n_i \in \mathbb{Z}_+$  for  $i \in I$ ,  $m_{si} \in \mathbb{Z}_+$  for  $1 \leq s \leq k$  and  $i \in I_0$ , and  $a_s \in \mathbb{R}_+$ , with  $\sum_{s=1}^k a_s = 1$ . Using the linear independence of  $\alpha_i$ ,  $i \in I$ , we see immediately that  $n_i = 0 \ \forall i \notin I_0$ , and hence  $\mu \in \lambda - Q_{I_0}^+$ . The reverse inclusion is proved, and we are done.  $\square$

**Remark.** In the case when  $\lambda \in P^+$  and  $J = J_\lambda = I$ , the Theorem is proved in [Vin]. Our proof as we mentioned in the introduction is quite different.

**2.7.** Another corollary of the above theorem is:

**Corollary.**  *$F$  is a face of  $\text{conv}_{\mathbb{R}}(\text{wt}(M(\lambda, J)))$  if and only if*

$$F = \text{conv}_{\mathbb{R}}(w(\text{wt } \mathbf{U}(\mathfrak{m}_{I_0 \cap J})m_\lambda)) - \text{cone}_{\mathbb{R}} w(\Phi_{I_0}^+ \setminus \Phi_{I_0 \cap J}^+)$$

for some  $w \in W_J$  and  $I_0 \subset I$ .

*Proof.* We first prove the following statement (which generalizes Proposition 2.4):

For all  $I_0, J \subset I$ ,

$$\text{conv}_{\mathbb{R}}(\text{wt } M(\lambda, J) \cap (\lambda - Q_{I_0}^+)) = \text{conv}_{\mathbb{R}}(\text{wt } \mathbf{U}(\mathfrak{m}_{I_0 \cap J})m_\lambda) - \text{cone}_{\mathbb{R}}(\Phi_{I_0}^+ \setminus \Phi_{I_0 \cap J}^+). \quad (2.1)$$

To prove this equation, note that  $\text{wt}(M(\lambda, J)) \cap (\lambda - Q_{I_0}^+)$  is the same as the set of weights of the  $\mathfrak{g}_{I_0}$ -submodule  $\mathbf{U}(\mathfrak{g}_{I_0})m_\lambda$ . Restricting our attention to  $\mathfrak{g}_{I_0}$ , we see that, as in Proposition 2.3,

$$\text{conv}_{\mathbb{R}}(\text{wt } \mathbf{U}(\mathfrak{g}_{I_0})m_\lambda) = \text{conv}_{\mathbb{R}}(\text{wt } \mathbf{U}(\mathfrak{m}_{I_0 \cap J})m_\lambda) - \left\{ \sum_{\alpha \in \Phi_{I_0}^+ \setminus \Phi_{I_0 \cap J}^+} r_\alpha \alpha : r_\alpha \in \mathbb{R}_+ \right\}.$$

This proves Equation (2.1). Now, by Theorem 1, if  $F$  is a face, there exist  $w \in W_J$  and  $I_0 \subset I$  such that  $w^{-1}(F) = \text{conv}_{\mathbb{R}}(\text{wt}(M(\lambda, J)) \cap (\lambda - Q_{I_0}^+))$ . The result then follows from Equation (2.1) and the linearity of  $w$ .  $\square$

We claim that the above corollary is a special case of the following more general result - which also generalizes a result of Vinberg in [Vin].

**Proposition.** *Let  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ , and let  $J \subset I$ . Suppose  $\mathbf{P}(\lambda, J)$  is the polyhedron in  $\mathfrak{h}_{\mathbb{R}}^*$  given by*

$$\mathbf{P}(\lambda, J) = \text{conv}_{\mathbb{R}}(W_J(\lambda)) - \text{cone}_{\mathbb{R}}(\Phi^+ \setminus \Phi_J^+).$$

*Then  $F$  is a face of  $\mathbf{P}(\lambda, J)$  if and only if*

$$w(F) = \text{conv}_{\mathbb{R}}(W_{J \cap I_0}(\lambda')) - \text{cone}_{\mathbb{R}}(\Phi_{I_0}^+ \setminus \Phi_{J \cap I_0}^+),$$

*for some  $w \in W_J$  and  $I_0 \subset I$ , where  $\lambda' \in W_J(\lambda)$  satisfies  $\lambda'(h_j) \geq 0 \ \forall j \in J$ .*

*Proof.* The proof goes through as in the proof of Theorem 1 once we note that if  $\mu \in \mathbf{P}(\lambda, J)$ , then  $\lambda' - \mu \in \mathbb{R}_+\Delta$ . For this, it suffices to show that  $\lambda' - w(\lambda) \in \mathbb{R}_+\Delta$  for all  $w \in W_J$ .

Consider the partial order on  $\mathfrak{h}_{\mathbb{R}}^*$  given by  $\mu \preceq \nu$  if and only if  $\nu - \mu \in \mathbb{R}_+\Delta$ . Recall that the intersection of the fundamental Weyl chamber and the Weyl orbit of any nonzero element in  $\mathfrak{h}_{\mathbb{R}}^*$  contains exactly one element. In particular, if  $w \in W_J$  and  $w(\lambda) \neq \lambda'$ , then  $w(\lambda)(h_j) < 0$  for some  $j \in J$ . Then,  $w(\lambda) \prec s_j(w(\lambda))$ .

Since the set  $W_J(\lambda)$  is finite, it must contain a maximal element with respect to the partial order. We have shown that  $w(\lambda) \neq \lambda'$  is not maximal, so  $\lambda'$  must be the unique maximal element in  $W_J(\lambda)$  in this partial order. In other words,  $\lambda' - w(\lambda) \in \mathbb{R}_+\Delta$  for all  $w \in W_J$ .  $\square$

### 3. RESULTS ON FINITE-DIMENSIONAL MODULES

Our other main results involve extending the notion of convexity and faces to arbitrary subfields  $\mathbb{F}$  of  $\mathbb{R}$ . We first note that for any  $X \subset \mathbb{R}^n$  and subfield  $\mathbb{F} \subset \mathbb{R}$ , we can define the  $\mathbb{F}$ -convex hull,  $\text{conv}_{\mathbb{F}}(X)$ , and  $\mathbb{F}$ -cone,  $\text{cone}_{\mathbb{F}}(X)$ , similar to the case when  $\mathbb{F} = \mathbb{R}$  in Section 2.4. Next, we extend the notion of relative interior as follows: the  $\mathbb{F}$ -relative interior of  $Y = \text{conv}_{\mathbb{F}}(X)$  is the subset

$$\text{relint}_{\mathbb{F}}(\text{conv}_{\mathbb{F}}(X)) = \{x \in Y \mid \forall y \in Y, \exists z \in Y, t \in \mathbb{F} \cap (0, 1) \text{ such that } x = ty + (1 - t)z\}.$$

It is clear that the  $\mathbb{R}$ -relative interior of a polyhedron does not intersect any proper face of the polyhedron.

**Remark.** For the remainder of the paper, we will freely use  $\text{relint}(\text{conv}_{\mathbb{F}}(X))$  to indicate the  $\mathbb{F}$ -relative interior. Strictly speaking, this is an abuse of notation: for example, if  $X$  is  $\mathbb{R}$ -convex in  $\mathbb{R}^n$ , then  $\text{relint}_{\mathbb{F}}(\text{conv}_{\mathbb{F}}(X)) = \text{relint}_{\mathbb{F}}(X)$  depends on  $\mathbb{F}$ . However, we only work with  $\text{relint}_{\mathbb{F}}(\text{conv}_{\mathbb{F}}(X))$  in this paper.

We now come to the two main new concepts in this paper. We are interested in studying certain subsets of sets  $X$ , that are related to the faces of  $\text{conv}_{\mathbb{R}}(X)$ . Among these, we further distinguish some of them.

**Definition.** Fix a subset  $X \subset \mathbb{R}^n$ , and a subfield  $\mathbb{F} \subset \mathbb{R}$ .

(i) We say that  $Y \subset X$  is a *weak  $\mathbb{F}$ -face* of  $X$  if for every  $U \subset X$ ,

$$\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}_{\mathbb{F}}(\text{conv}_{\mathbb{F}}(U)) \neq \emptyset \implies U \subset Y.$$

(ii) A weak  $\mathbb{F}$ -face  $Y \subset X$  is a *positive weak  $\mathbb{F}$ -face* of  $X$  if for every  $U \subset X$ ,

$$\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}_{\mathbb{F}}(\text{conv}_{\mathbb{F}}(U \cup \{0\})) = \emptyset.$$

(Clearly,  $X$  is always a weak  $\mathbb{F}$ -face of  $X$ .) As we will see below, (positive) weak  $\mathbb{F}$ -faces of  $X$  are closely related to the faces of  $\text{conv}_{\mathbb{R}}(X)$ , when the latter is a polyhedron. Our main results now characterize the (positive) weak  $\mathbb{F}$ -faces of (the set of weights of) finite-dimensional  $\mathfrak{g}$ -modules.

**Theorem 2.** *Suppose  $V$  is a finite-dimensional  $\mathfrak{g}$ -module, and  $\mathbb{F}$  is a subfield of  $\mathbb{R}$ . Then either  $\text{wt } V = \{0\}$ , or the following are equivalent for a proper subset  $Y \subset \text{wt } V$ :*

- (i)  $Y$  is a positive weak  $\mathbb{F}$ -face of  $\text{wt } V$ .
- (ii)  $Y$  is a weak  $\mathbb{F}$ -face of  $\text{wt } V$ .
- (iii)  $Y = F \cap \text{wt } V$ , for some proper face  $F$  of  $\text{conv}_{\mathbb{R}}(\text{wt } V)$ .

As we see in Lemma 4.2, faces of the polytope  $\text{conv}_{\mathbb{R}}(\text{wt } V)$  are precisely maximizers of linear functionals; thus, our result generalizes a result in [CDR, CG], which was stated only for the simple module  $V = \mathfrak{g}$ , and proved using a case-by-case analysis involving long and short roots.

Our next result is a characterization of precisely which subsets of  $\text{wt } V(\lambda)$  form (positive) weak  $\mathbb{F}$ -faces, and once again, it generalizes (and recovers) the example of  $V(\lambda) = \mathfrak{g}$  that was studied in [CDR]. Moreover, it combines features from both the theorems above (Theorems 1 and 2).

**Theorem 3.** *Suppose  $\mathbb{F} \subset \mathbb{R}$ ,  $0 \neq \lambda \in P^+$ , and  $V(\lambda) = M(\lambda, I)$  is simple. Then the following are equivalent for a subset  $Y \subset \text{wt } V(\lambda)$ :*

- (i) There exist  $w \in W$  and  $I_0 \subset I$  such that  $wY = \text{wt } V(\lambda) \cap (\lambda - Q_{I_0}^+)$ .
- (ii)  $Y$  is a weak  $\mathbb{F}$ -face of  $\text{wt } V(\lambda)$ .
- (iii) Let  $\rho_Y := \sum_{y \in Y} y$ . Then  $Y$  is the maximizer in  $\text{wt } V(\lambda)$  of the linear functional  $(\rho_Y, -)$ .

If, furthermore,  $Y \neq \text{wt } V(\lambda)$ , then  $\rho_Y \in P^+$  and the functional  $(\rho_Y, -)$  has positive maximum on  $\text{wt } V(\lambda)$ .

Note that both of these results (Theorems 2 and 3) are independent of  $\mathbb{F}$ . Moreover, the vector  $\rho_Y$  has a geometric interpretation: it is a positive rational multiple of the “center of mass” of the face  $\text{conv}_{\mathbb{R}}(Y)$  of  $\text{conv}_{\mathbb{R}}(\text{wt } V(\lambda))$ .

To state our last result, we need two more pieces of notation.

**Definition.** Given  $\lambda \in \mathfrak{h}^*$  and  $I_0 \subset I$ , define  $\text{wt } V_{I_0}(\lambda) := \text{wt } V(\lambda) \cap (\lambda - Q_{I_0}^+)$ , and  $\rho_{\lambda, I_0} := \rho_{\text{wt } V_{I_0}(\lambda)}$ .

We now answer a natural question arising from Vinberg’s result.

**Theorem 4.** *If  $0 \neq \lambda \in P^+$ , then for any  $I_1, I_2 \subset I$ ,  $\text{wt } V_{I_1}(\lambda) = \text{wt } V_{I_2}(\lambda)$  if and only if  $\rho_{\lambda, I_1} = \rho_{\lambda, I_2}$ , if and only if the sets of vertices coincide:  $W_{I_1}(\lambda) = W_{I_2}(\lambda)$ .*



#### 4. FACES OF POLYHEDRA

Our main goal in this section is to develop the techniques that will be needed to prove the theorems in Section 3. In particular, we will show the following result.

**Theorem.** *Suppose  $\text{conv}_{\mathbb{R}}(X)$  is a polyhedron for  $X \subset \mathbb{F}^n$ . Then  $Y \subset X$  is a weak  $\mathbb{F}$ -face if and only if  $Y$  maximizes a linear functional in  $X$ . If instead,  $\text{conv}_{\mathbb{R}}(X \cup \{0\})$  is a polyhedron for  $X \subset \mathbb{F}^n$ , then  $Y \subset X$  is a positive weak  $\mathbb{F}$ -face if and only if  $Y$  maximizes a linear functional in  $X$ , and this maximum value is positive.*

This result also explains the choice of terminology behind (positive) weak  $\mathbb{F}$ -faces.

**4.1.** The following lemma will be crucial in our examination of these sets.

**Lemma.** *Suppose  $X \subset \mathbb{F}^n$ . If  $u$  is in the  $\mathbb{F}$ -relative interior of  $\text{conv}_{\mathbb{F}}(X)$  and  $x_0 \in X$ , then there exist  $m > 0$ ,  $r_0, r_1, \dots, r_m \in \mathbb{F}_{>0}$ , and  $x_1, \dots, x_m \in X$  such that*

$$u = r_0 x_0 + \sum_{j=1}^m r_j x_j, \quad r_0 + \sum_{j=1}^m r_j = 1.$$

*Proof.* Since  $u$  is in the interior of  $\text{conv}_{\mathbb{F}}(X)$ , we can find  $t \in \mathbb{F}_{>0}$  such that  $u = tx_0 + (1-t)x'$ , where  $x' \in \text{conv}_{\mathbb{F}}(X)$ . By definition of  $\text{conv}_{\mathbb{F}}(X)$ , we can write  $x' = \sum_{j=1}^m s_j x_j$  for some  $x_j \in X$  and  $s_j \in \mathbb{F}_{>0}$  such that  $\sum_{j=1}^m s_j = 1$ . Solving for  $u$ , we have

$$u = tx_0 + \sum_{j=1}^m s_j(1-t)x_j.$$

Setting  $r_0 = t$  and  $r_j = s_j(1-t)$  gives the result.  $\square$

We remark that if  $X$  is not a singleton, we may choose all  $x_0, x_1, \dots, x_m$  to be distinct from  $u$ : we start by choosing any  $x_0 \neq u$  in  $\text{conv}_{\mathbb{F}}(X)$ , and proceed as above. Now if  $x_j = u$  for some  $j > 0$ , then we simply subtract  $r_j u$  from both sides, and divide by  $1 - r_j$ .

**4.2.** The following lemma will also be used frequently.

**Lemma.** *Let  $\mathcal{P} \subset \mathbb{R}^n$  be nonempty. A nonempty subset  $F \subset \mathcal{P}$  is a face of  $\mathcal{P}$  if and only if  $F$  is the subset of  $\mathcal{P}$  that maximizes some linear functional  $\varphi \in (\mathbb{R}^n)^*$ .*

*Proof.* If  $F$  is a face of  $\mathcal{P}$ ,  $F = \mathcal{P} \cap H(v, w)$  for some supporting hyperplane  $H(v, w)$ . Define  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $\varphi(u) = -v \cdot u$ . It is easy to see that  $\varphi$  is maximized in  $\mathcal{P}$  precisely on  $F$ .

Similarly, if  $\varphi \in (\mathbb{R}^n)^*$  is maximized in  $\mathcal{P}$  on  $F$ , choose  $v$  such that  $\varphi(u) = -v \cdot u$  for all  $u \in \mathbb{R}^n$ . Let  $w \in F$ . Then,  $F = \mathcal{P} \cap H(v, w)$ .  $\square$

**Proposition.** *Suppose that  $X \subset \mathbb{F}^n$ . Then  $\text{conv}_{\mathbb{F}}(X) = \text{conv}_{\mathbb{R}}(X) \cap \mathbb{F}^n$ .*

*Proof.* The inclusion  $\text{conv}_{\mathbb{F}}(X) \subset \text{conv}_{\mathbb{R}}(X) \cap \mathbb{F}^n$  is obvious.

Suppose that  $u \in \text{conv}_{\mathbb{R}}(X) \cap \mathbb{F}^n$ . Then,  $u \in \text{conv}_{\mathbb{R}}(U)$  for some finite subset  $U \subset X$ . By Caratheodory's theorem,  $u$  is in some  $r$ -simplex  $S \subset \text{conv}_{\mathbb{R}}(U)$ , such that the vertices of  $S$  are a subset of  $U$ .

Let  $\{s_0, s_1, \dots, s_r\}$  be the vertices of  $S$ . Then,  $u = \sum_{i=0}^r n_i s_i$  for some  $n_i \in \mathbb{R}_+$  with  $\sum_{i=0}^r n_i = 1$ .

Let  $\psi$  be an  $\mathbb{F}$ -affine transformation of  $\mathbb{R}^n$  such that  $\psi(s_0) = \mathbf{0}$  and  $\psi(s_i) = \mathbf{e}_i$ ,  $1 \leq i \leq r$ , where  $\{\mathbf{e}_i : 1 \leq i \leq n\}$  are the standard basis vectors in  $\mathbb{R}^n$ . It is easy to see that

$$\psi(u) = \sum_{i=1}^r n_i \mathbf{e}_i.$$

Since  $\psi$  is  $\mathbb{F}$ -affine,  $\psi(u) \in \mathbb{F}_+^n$ , and  $n_i \in \mathbb{F}_+$  for  $i > 0$ . Furthermore,  $n_0 = 1 - \sum_{i=1}^r n_i \in \mathbb{F}_+$ , so  $u \in \text{conv}_{\mathbb{F}}(X)$ .  $\square$

**Corollary.** *Suppose  $X \subset \mathbb{F}^n$ , and  $F$  is a face of  $\text{conv}_{\mathbb{R}}(X)$ . Then  $F \cap \mathbb{F}^n$  is a face of  $\text{conv}_{\mathbb{F}}(X)$ .*

*Proof.* Let  $H(v, w)$  be a supporting hyperplane for  $\text{conv}_{\mathbb{R}}(X)$  such that  $F = \text{conv}_{\mathbb{R}}(X) \cap H(v, w)$ . Then,

$$F \cap \mathbb{F}^n = \text{conv}_{\mathbb{R}}(X) \cap H(v, w) \cap \mathbb{F}^n = \text{conv}_{\mathbb{F}}(X) \cap H(v, w).$$

$\square$

**4.3.** We now prove a general result relating weak  $\mathbb{F}$ -faces and polyhedra.

**Theorem.** *Suppose that  $\text{conv}_{\mathbb{R}}(X)$  is a polyhedron for  $X \subset \mathbb{F}^n$ . Then,  $Y \subset X$  is a weak  $\mathbb{F}$ -face if and only if  $Y = F \cap X$  for some face  $F$  of  $\text{conv}_{\mathbb{R}}(X)$ . Moreover,  $\text{conv}_{\mathbb{R}}(Y) = F$  in this case.*

In particular, faces of polyhedra are weak  $\mathbb{R}$ -faces, using  $\mathbb{F} = \mathbb{R}$ .

*Proof.* First, suppose that  $Y = F \cap X$  for some face  $F$  of  $\text{conv}_{\mathbb{R}}(X)$ . By Lemma 4.2, one can find a linear functional  $\varphi \in (\mathbb{R}^n)^*$  such that  $\varphi(u) \geq \varphi(v)$  for all  $u \in F$  and  $v \in \text{conv}_{\mathbb{R}}(X)$ . Let  $x_0 \in U \subset X$ , and suppose  $u \in \text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U))$ .

We can write  $u = \sum_{y \in Y} s_y y$  with  $s_y \in \mathbb{F}_+$  and  $\sum_{y \in Y} s_y = 1$ , and, thus,  $\varphi(u) = \varphi(F)$ . By Lemma 4.1,  $u = \sum_{j=0}^m r_j x_j$  for some  $r_j \in \mathbb{F}_{>0}$  and  $x_j \in U$ . Applying  $\varphi$ , we have

$$\varphi(F) = \varphi(u) = \sum_{j=0}^m r_j \varphi(x_j) \leq \sum_{j=0}^m r_j \varphi(F) = \varphi(F).$$

Since  $r_0$  is positive,  $\varphi(x_0) = \varphi(F)$ , so  $x_0 \in F \cap X = Y$ . Since  $x_0 \in U$  was arbitrary,  $U \subset Y$ , so  $Y = F \cap X$  is a weak  $\mathbb{F}$ -face of  $X$ .

Now, let  $Y$  be a weak  $\mathbb{F}$ -face of  $X$ , and let  $F$  be the smallest face of  $\text{conv}_{\mathbb{R}}(X)$  such that  $Y \subset F$ . If  $\#F \cap X = 1$ , then  $Y = F \cap X$  and we are done. Suppose that  $\#F \cap X > 1$ . Since  $F$  is minimal and  $\text{conv}_{\mathbb{R}}(X)$  is a polyhedron, the interior of  $F$  must contain an element  $y \in \text{conv}_{\mathbb{F}}(Y)$ . Let  $x \in F \cap X$ . If  $x = y$ , then it is clear that  $x \in Y$ .

Suppose that  $x \neq y$ . Then, by Lemma 4.1,  $y = r_0 x + \sum_{i=1}^m r_i x_i$  for some  $r_i \in \mathbb{F}_{>0}$  and  $x_i \in F \cap X$ . In particular,  $y \in \text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(F \cap X))$ . Since  $Y$  is a weak  $\mathbb{F}$ -face of  $X$ , this gives that  $F \cap X \subset Y$ .

Finally, given  $Y = F \cap X$  for some face  $F$  of  $\text{conv}_{\mathbb{R}}(X)$ , clearly we have  $\text{conv}_{\mathbb{R}}(Y) \subset F$ . Conversely, given  $f \in F \subset \text{conv}_{\mathbb{R}}(X)$ ,  $1 \cdot f = \sum_i a_i x_i$  for some  $x_i \in X$ , with  $0 \leq a_i$  adding up

to 1. Now use Proposition 4.4 with  $\mathbb{F} = \mathbb{R}$ : since  $F$  is a weak  $\mathbb{R}$ -face of the polyhedron (by the remark following the statement of this result), hence each  $x_i \in F$ . But then  $x_i \in F \cap X = Y$ , so  $f \in \text{conv}_{\mathbb{R}}(Y)$  as desired.  $\square$

**4.4.** We now study positive weak  $\mathbb{F}$ -faces. We start with an equivalent characterization.

**Lemma.** *For all subsets  $X \subset \mathbb{R}^n$  and subfields  $\mathbb{F} \subset \mathbb{R}$ , the positive weak  $\mathbb{F}$ -faces of  $X$  are the weak  $\mathbb{F}$ -faces  $Y \subset X$  such that  $Y$  is a weak  $\mathbb{F}$ -face of  $X \cup \{0\}$  and  $0 \notin \text{conv}_{\mathbb{F}}(Y)$ .*

*Proof.* First, suppose that  $Y$  is a positive weak  $\mathbb{F}$ -face of  $X$ . It follows easily from the definition that  $Y$  is a weak  $\mathbb{F}$ -face of  $X \cup \{0\}$ . Suppose that  $0 \in \text{conv}_{\mathbb{F}}(Y)$ , and let  $U = Y$ . Then, it is clear that  $\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U \cup \{0\})) = \text{relint}(\text{conv}_{\mathbb{F}}(Y)) \neq \emptyset$ , which contradicts  $Y$  being a positive weak  $\mathbb{F}$ -face of  $X$ .

Now, suppose  $Y$  is a weak  $\mathbb{F}$ -face for both  $X$  and  $X \cup \{0\}$  such that  $0 \notin \text{conv}_{\mathbb{F}}(Y)$ . Let  $U \subset X$ . If  $\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U \cup \{0\})) \neq \emptyset$ , then  $U \cup \{0\} \subset Y$  since  $Y$  is a weak  $\mathbb{F}$ -face of  $X \cup \{0\}$ . However, this is impossible since  $0 \notin \text{conv}_{\mathbb{F}}(Y)$ . Thus,  $Y$  is a positive weak  $\mathbb{F}$ -face of  $X$ .  $\square$

To connect these results to the results in [CKR], we prove the following proposition.

**Proposition.** *Let  $X \subset \mathbb{R}^n$  and  $\mathbb{F}$  a subfield of  $\mathbb{R}$ .*

(i) *A subset  $Y$  is a weak  $\mathbb{F}$ -face of  $X$  if and only if*

$$\sum_{y \in Y} m_y y = \sum_{x \in X} r_x x, \quad m_y, r_x \in \mathbb{F}_+ \quad \forall y \in Y, x \in X \quad \text{and} \quad \sum_{y \in Y} m_y = \sum_{x \in X} r_x \implies x \in Y \text{ if } r_x \neq 0.$$

(ii) *A subset  $Y$  is a positive weak  $\mathbb{F}$ -face of  $X$  if and only if (i) holds and*

$$\sum_{y \in Y} m_y y = \sum_{x \in X} r_x x \implies \sum_{y \in Y} m_y \leq \sum_{x \in X} r_x.$$

*Proof.* (i)  $(\Leftarrow)$  Suppose  $U \subset X$  and  $u \in \text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U))$ . Let  $x_0 \in U$ . By Lemma 4.1 and the definition of  $\text{conv}_{\mathbb{F}}(Y)$ , we can write  $u = \sum_{y \in Y} m_y y = r_0 x_0 + \sum_{j=1}^m r_j x_j$  for some  $x_j \in U$ , where  $m_y \in \mathbb{F} \cap [0, 1]$  for all  $y \in Y$ ,  $r_j \in \mathbb{F} \cap (0, 1)$  for  $j = 0, \dots, m$ , and  $\sum_{y \in Y} m_y = \sum_{j=0}^m r_j = 1$ . Then,  $r_0 \neq 0$ , so  $x_0 \in Y$ . Since  $x_0$  was arbitrary,  $U \subset Y$ .

$(\Rightarrow)$  Suppose that  $u = \sum_{y \in Y} m_y y = \sum_{x \in X} r_x x$  and  $\sum_{y \in Y} m_y = \sum_{x \in X} r_x > 0$  with  $m_y, r_x \in \mathbb{F}_+$  for all  $y \in Y$  and  $x \in X$ .

Let  $U = \{x \in X \mid r_x \neq 0\}$ , and consider  $u' = \frac{1}{\sum_{x \in X} r_x} u \in \text{conv}_{\mathbb{F}}(U)$ . It is clear that  $u' \in \text{conv}_{\mathbb{F}}(Y)$ , so it suffices to show that  $u' \in \text{relint}(\text{conv}_{\mathbb{F}}(U))$ . Furthermore, since each  $x \in \text{conv}_{\mathbb{F}}(U)$  is a convex sum of a finite number of elements in  $U$ , it suffices to check that for every  $x_0 \in U$ , there exists  $y_0 \in \text{conv}_{\mathbb{F}}(U)$  and  $r_0 \in \mathbb{F} \cap (0, 1)$  such that  $u' = r_0 x_0 + (1 - r_0) y_0$ . By construction, we have  $u' = \sum_{x \in U} r'_x x$  with  $r'_x \in \mathbb{F} \cap (0, 1)$  and  $\sum_{x \in U} r'_x = 1$ . Letting  $r_0 = r'_{x_0}$ , we have

$$u' = r_0 x_0 + \sum_{x \neq x_0} r'_x x = r_0 x_0 + (1 - r_0) \sum_{x \neq x_0} \frac{r'_x}{1 - r_0} x.$$

It is easy to check that  $y = \sum_{x \neq x_0} \frac{r'_x}{1-r_0} x \in \text{conv}_{\mathbb{F}}(U)$ . In particular,  $\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U)) \neq \emptyset$ , so  $U = \{x \in X \mid r_x \neq 0\} \subset Y$ .

- (ii) ( $\Leftarrow$ ) Suppose that  $\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U \cup \{0\})) \neq \emptyset$  for some  $U \subset X$ . Let  $u \in \text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U \cup \{0\}))$ . By Lemma 4.1, there exist  $r_0, r_1, \dots, r_n \in \mathbb{F} \cap (0, 1)$  and  $x_1, \dots, x_n \in U$  such that  $\sum_{j=1}^n r_j = 1$ , and

$$u = r_0 \cdot 0 + \sum_{j=1}^n r_j x_j = \sum_{j=1}^n r_j x_j.$$

Similarly, there exist  $m_y \in \mathbb{F} \cap [0, 1]$  such that  $\sum_{y \in Y} m_y = 1$ , and  $u = \sum_{y \in Y} m_y y$ . However, this gives  $u = \sum_{y \in Y} m_y y = \sum_{j=1}^n r_j x_j$  with  $\sum_{j=1}^n r_j = 1 - r_0 < 1 = \sum_{y \in Y} m_y$ , which is impossible. Thus,  $\text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U \cup \{0\})) = \emptyset$  for all  $U \subset X$ .

( $\Rightarrow$ ) Let  $u = \sum_{y \in Y} m_y y = \sum_{x \in X} r_x x$  with  $m_y, r_x \in \mathbb{F} \cap [0, \infty)$ . Let  $U = \{x \in X \mid r_x \neq 0\} \neq \emptyset$ , and suppose that  $\sum_{x \in X} r_x < \sum_{y \in Y} m_y$ . Define  $r_0 = \sum_{y \in Y} m_y - \sum_{x \in X} r_x > 0$ . Then,  $u = r_0 \cdot 0 + \sum_{x \in U} r_x x$ .

Let  $u' = \frac{1}{\sum_{y \in Y} m_y} u$ . Since  $r_0 + \sum_{x \in U} r_x = \sum_{y \in Y} m_y$ ,  $u' \in \text{conv}_{\mathbb{F}}(U \cup \{0\})$ . In fact, by an argument similar to that in part (i),  $u' \in \text{relint}(\text{conv}_{\mathbb{F}}(U \cup \{0\}))$ . However, this gives  $u' \in \text{conv}_{\mathbb{F}}(Y) \cap \text{relint}(\text{conv}_{\mathbb{F}}(U \cup \{0\}))$ , which contradicts  $Y$  being a positive weak  $\mathbb{F}$ -face of  $X$ . Therefore,  $\sum_{y \in Y} m_y \leq \sum_{x \in X} r_x$ . □

Finally, this equivalent formulation of the positive weak  $\mathbb{F}$ -faces allows us to explain the terminology.

**Theorem.** Suppose  $\text{conv}_{\mathbb{R}}(X \cup \{0\})$  is a polyhedron for  $X \subset \mathbb{F}^n$ . Then  $Y \subset X$  is a positive weak  $\mathbb{F}$ -face of  $X$  if and only if  $Y$  maximizes in  $X$  some linear functional  $\varphi \in (\mathbb{R}^n)^*$  which has a positive maximum on  $X$ .

In particular, if  $0$  is in the interior of  $\text{conv}_{\mathbb{R}}(X)$ , then a subset  $Y$  is a positive weak  $\mathbb{F}$ -face of  $X$  if and only if  $Y \neq X$  and  $Y$  is a weak  $\mathbb{F}$ -face of  $X$ .

*Proof.* If  $Y$  is a positive weak  $\mathbb{F}$ -face of  $X$ , then  $Y$  is a positive weak  $\mathbb{F}$ -face of  $X \cup \{0\}$ , and hence also a weak  $\mathbb{F}$ -face of  $X \cup \{0\}$  (both statements follow from the definitions), which does not contain  $0$  by Lemma 4.4. Hence by Theorem 4.3,  $Y = F \cap (X \cup \{0\}) = F \cap X$ , for some face  $F$  of  $\text{conv}_{\mathbb{R}}(X \cup \{0\})$ . Suppose  $F$  maximizes the linear functional  $\varphi$  in the polyhedron. Now if  $0 \in F$ , then  $0 \in F \cap (X \cup \{0\}) = Y$ , which contradicts Lemma 4.4. Thus,  $Y = F \cap X$  maximizes  $\varphi$  in  $X \cup \{0\}$ , and  $0 \notin F$ . Hence  $\varphi(Y) > \varphi(0) = 0$ .

Conversely, choose  $\varphi \in (\mathbb{R}^n)^*$  which is maximized in  $X$  precisely on  $Y$  and  $\varphi(Y) > 0$ . (In particular,  $Y$  is a weak  $\mathbb{F}$ -face by Theorem 4.3 and Lemma 4.2.) Suppose that  $\sum_{y \in Y} m_y y = \sum_{x \in X} r_x x$ . Applying  $\varphi$ , we have

$$\varphi(Y) \sum_{y \in Y} m_y = \sum_{y \in Y} m_y \varphi(y) = \sum_{x \in X} r_x \varphi(x) \leq \varphi(Y) \sum_{x \in X} r_x.$$

Since  $\varphi(Y) > 0$ , this gives  $\sum_{y \in Y} m_y \leq \sum_{x \in X} r_x$ , and  $Y$  is a positive weak  $\mathbb{F}$ -face of  $X$  by Proposition 4.4.

Finally, suppose that  $0 \in \text{relint}_{\mathbb{R}}(\text{conv}_{\mathbb{R}}(X))$ . The result is clear if  $X = \{0\}$ , so now suppose otherwise. Since 0 is an interior point,  $0 \in \text{conv}_{\mathbb{R}}(X \setminus \{0\}) \cap \mathbb{F}^n$ , so by Proposition 4.2,  $0 \in \text{conv}_{\mathbb{F}}(X \setminus \{0\}) \subset \text{conv}_{\mathbb{F}}(X)$ . Then,  $X$  is not a positive weak  $\mathbb{F}$ -face of itself, by Lemma 4.4. Since every positive weak  $\mathbb{F}$ -face is a weak  $\mathbb{F}$ -face, it now suffices to prove that every proper weak  $\mathbb{F}$ -face of  $X$  is a positive weak  $\mathbb{F}$ -face.

Let  $Y \subsetneq X$  be a (proper) weak  $\mathbb{F}$ -face of  $X$ . By Theorem 4.3,  $Y = F \cap X$ , for some proper face  $F$  of  $\text{conv}_{\mathbb{R}}(X) = \text{conv}_{\mathbb{R}}(X \cup \{0\})$ . Since 0 is an interior point,  $0 \notin F$ , so  $0 \notin Y = F \cap (X \cup \{0\})$ . By Lemma 4.2,  $Y \subset X$  maximizes some linear functional  $\varphi$  on  $X \cup \{0\}$ , and  $0 \notin Y$ . Hence  $\varphi(Y) > \varphi(0) = 0$ , and we are done by the first part of this result.  $\square$

## 5. APPLICATION TO REPRESENTATION THEORY

We can now show one of our main results, using the above theory.

*Proof of Theorem 2.* Suppose  $\text{wt } V \neq \{0\}$ . By the Decomposition Theorem 2.5, the sets  $\text{conv}_{\mathbb{R}}(\text{wt } V)$  and  $\text{conv}_{\mathbb{R}}(\{0\} \cup \text{wt } V)$  are polytopes. The result follows from Theorem 4.3 and Theorem 4.4, once we show that the origin is in the  $\mathbb{F}$ -relative interior of  $\text{conv}_{\mathbb{F}}(\text{wt } V)$ , for all  $\mathbb{F}$ .

First note that the vector  $\rho_V := \sum_{\mu \in \text{wt } V} \mu$  is  $W$ -invariant, since  $\text{wt } V$  is stable under  $W$ . Then  $s_i(\rho_V) = \rho_V$ , so  $(\rho_V, \alpha_i) = 0 \ \forall i$ . Thus,  $\rho_V = 0$ . Now given  $y = \sum_{\mu \in \text{wt } V} r_{\mu} \mu \in \text{conv}_{\mathbb{F}}(\text{wt } V)$ , define

$$z = \frac{1}{|\text{wt } V| - 1} \sum_{\mu \in \text{wt } V} (1 - r_{\mu}) \mu \in \text{conv}_{\mathbb{F}}(\text{wt } V).$$

Then  $\rho_V = 0 = ty + (1-t)z$ , where  $t = \frac{1}{|\text{wt } V|} \in \mathbb{F} \cap (0, 1)$ . Hence  $0 = \rho_V \in \text{relint}(\text{conv}_{\mathbb{F}}(\text{wt } V))$ .  $\square$

**5.1.** We now prove the following result, before using it to show Theorem 3. We introduce the following notation: given  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ , define  $I_{\lambda}$  to be the union of those graph components of the Dynkin diagram of  $\mathfrak{g}$ , which are not disjoint from  $\text{supp}(\lambda)$ .

**Proposition.** Fix  $0 \neq \lambda \in P^+$  and  $J \subset I$ . Then  $\text{wt } V_J(\lambda) \subsetneq \text{wt } V(\lambda)$  if and only if  $I_{\lambda} \not\subseteq J$ , if and only if  $\max_{\mu \in \text{wt } V(\lambda)} \rho_{\lambda, J} > 0$ . (Hereafter, we abuse notation, whereby  $\mu \in \mathfrak{h}_{\mathbb{R}}^*$  denotes the functional  $(\mu, -)$ .)

*Proof.* We first make the following claim:

$$\text{wt } V_J(\lambda) = \text{wt } V_{J \cap I_{\lambda}}(\lambda). \quad (5.1)$$

Let us show the claim first. Clearly  $\text{wt } V_{J \cap I_{\lambda}}(\lambda) \subset \text{wt } V_J(\lambda)$ . Next, suppose  $\mu = \lambda - \sum_i a_i \alpha_i$  is any weight of  $V(\lambda)$ . Then there is some  $f \in U(\mathfrak{n}^-)_{\mu - \lambda}$  such that  $f v_{\lambda}$  is a nonzero weight vector. Since  $U(\mathfrak{n}^-)$  is the subalgebra of  $U(\mathfrak{g})$  generated by the  $x_{\alpha_i}^-$  ( $i \in I$ ), write  $f$  as a  $\mathbb{C}$ -linear combination of monomial words (each of weight  $\mu - \lambda$ ). Then at least one such monomial word  $x_{\alpha_{i_k}}^- \dots x_{\alpha_{i_2}}^- x_{\alpha_{i_1}}^-$  does not kill any highest weight vector  $0 \neq v_{\lambda} \in V(\lambda)_{\lambda}$ .

The claim is proved if we show that  $a_i = 0 \ \forall i \notin I_\lambda$ . Suppose not. Then there exists  $1 \leq j \leq k$  such that  $i_j \notin I_\lambda$ . Choose the minimal such  $j$ . Also note that  $x_{\alpha_{i_j}}^- \dots x_{\alpha_{i_1}}^- v_\lambda \neq 0$ . Now since  $x_{\alpha_{i_j}}^-$  commutes with  $x_{\alpha_{i_l}}^-$  for all  $0 < l < j$  (by the defining relations), we get:

$$x_{\alpha_{i_{j-1}}}^- \dots x_{\alpha_{i_1}}^- (x_{\alpha_{i_j}}^- v_\lambda) \neq 0,$$

whence  $x_{\alpha_{i_j}}^- v_\lambda \neq 0$ . Then, this is a nonzero weight vector in the simple module  $V(\lambda)$  of weight  $\lambda - \alpha_{i_j} \neq \lambda$ , so this vector cannot be maximal either; i.e., it is not killed by all of  $\mathfrak{n}^+$ . Now  $\mathfrak{n}^+$  is generated by  $\{x_{\alpha_i}^+ : i \in I\}$ . For  $i \neq i_j$ ,  $x_{\alpha_i}^+$  commutes with  $x_{\alpha_{i_j}}^-$ , so

$$x_{\alpha_i}^+ (x_{\alpha_{i_j}}^- v_\lambda) = x_{\alpha_{i_j}}^- \cdot x_{\alpha_i}^+ v_\lambda = 0.$$

Hence we must have:  $x_{\alpha_{i_j}}^+ \cdot x_{\alpha_{i_j}}^- v_\lambda \neq 0$ . The left-hand side equals  $\lambda(h_{\alpha_{i_j}})v_\lambda$  by standard computations, so  $\lambda(h_{\alpha_{i_j}}) \neq 0$ . However, this is a contradiction since  $\lambda(h_{\alpha_i}) = 0$  for all  $i \notin I_\lambda$ . Thus the claim is proved, and  $a_i = 0 \ \forall i \notin I_\lambda$ .

We are now ready to prove the result. We first show two of the cyclic implications (more precisely, we show their contrapositives). If  $J \supset I_\lambda$ , then by (5.1),

$$\text{wt } V_J(\lambda) = \text{wt } V_{J \cap I_\lambda}(\lambda) = \text{wt } V_{I_\lambda}(\lambda) = \text{wt } V_I(\lambda) = \text{wt } V(\lambda),$$

and we are done. Next,  $\rho_{\lambda, I} = \rho_V = 0$  (see the proof of Theorem 2), so we have:  $\max_{\text{wt } V(\lambda)} \rho_{\lambda, I} = \max 0 = 0$ .

Finally, suppose  $I_\lambda \not\subseteq J$ ; we prove that  $\max_{\text{wt } V(\lambda)} \rho_{\lambda, J} > 0$ . Since each weight is in  $\lambda - Q^+$ ,  $\rho_{\lambda, J} = |\text{wt } V_J(\lambda)|\lambda - \sum_{j \in J_1} m_j \alpha_j$  for some positive integers  $m_j$  and some subset  $J_1 \subset J$ . Since  $I_\lambda \not\subseteq J$ , there exists a graph component  $I_j \subset I_\lambda$  in the Dynkin diagram for  $\mathfrak{g}$ , such that  $I_j \not\subseteq J$ . We first show the following

**Claim.** There exists  $j_0 \in I_j \subset I_\lambda$ , such that  $(\rho_{\lambda, J}, \alpha_{j_0}) > 0$ .

*Proof.* We have two cases. First, suppose that  $I_j \cap J_1 = \emptyset$ . Since  $\text{supp}(\lambda) \cap I_j \neq \emptyset$ , choose  $j_0 \in I_j$  such that  $(\lambda, \alpha_{j_0}) > 0$ . Now since  $J_1 \cap I_j = \emptyset$ , we also have  $(\alpha_i, \alpha_{j_0}) = 0 \ \forall i \in J_1$ . Then  $(\rho_{\lambda, J}, \alpha_{j_0}) = |\text{wt } V_J(\lambda)|(\lambda, \alpha_{j_0}) > 0$ .

On the other hand, if  $I_j \cap J_1 \neq \emptyset$ , then since  $I_j$  is connected, choose  $j_0 \in I_j \setminus J_1$ , that is adjacent to at least one element  $i_0 \in J_1$ . Now

$$(\rho_{\lambda, J}, \alpha_{j_0}) = |\text{wt } V_J(\lambda)|(\lambda, \alpha_{j_0}) - \sum_{j \in J} m_j (\alpha_j, \alpha_{j_0}) \geq 0 + \sum_{j \in J_1 \setminus \{i_0\}} m_j \cdot 0 - m_{i_0} (\alpha_{i_0}, \alpha_{j_0}),$$

and this is strictly positive because  $i_0, j_0$  are connected by an edge in  $I_j$ .  $\square$

Returning to the proof of the result, since  $\lambda = \sum_{i \in \text{supp}(\lambda)} (\lambda, \alpha_i) \omega_i$ ,  $\lambda = \sum_{i \in I_\lambda} a_i \alpha_i$  with all  $a_i \in \mathbb{Q}_{>0}$  by [Hum, Exercise 13.8]. We now compute:

$$\max_{\text{wt } V(\lambda)} \rho_{\lambda, J} \geq (\rho_{\lambda, J}, \lambda) = \sum_{i \in I_\lambda} a_i (\rho_{\lambda, J}, \alpha_i) \geq a_{j_0} (\rho_{\lambda, J}, \alpha_{j_0}) > 0.$$

$\square$

We now show another of our main results. We need some more notation.

**Definition.** Define, for any  $J \subset I$ ,

$$\Delta_J := \{\alpha_j : j \in J\}, \quad \Delta := \Delta_I, \quad \Omega_J := \{\omega_j : j \in J\}, \quad \Omega := \Omega_I.$$

Now given  $X \subset \mathfrak{h}_{\mathbb{R}}^*$ , define  $X(\lambda)$  to be:

$$X(\lambda) := \{x \in X : (x, \lambda) \geq (x', \lambda) \ \forall x' \in X\} \subset X.$$

**Remark.** It is not hard to show that  $X(\lambda)$  is a weak  $\mathbb{F}$ -face of  $X$  for all  $\lambda$  and all  $\mathbb{F}$ , and that if  $\lambda(x) > 0$  for some  $x \in X$ , then  $X(\lambda)$  is a positive weak  $\mathbb{F}$ -face.

*Proof of Theorem 3.* By Theorem 2 and Lemma 4.2, (iii)  $\implies$  (ii). By Theorem 1 for  $J = J_\lambda = I$ , (ii)  $\implies$  (i). It remains to show that (i)  $\implies$  (iii) (and the second part of the theorem). Since  $w$  acts linearly on  $\mathfrak{h}_{\mathbb{R}}^*$  and  $(\cdot, \cdot)$  is  $W$ -invariant, it suffices to prove that (i)  $\implies$  (iii) for  $w = 1$ .

We now show that  $\text{wt } V_J(\lambda) = (\text{wt } V(\lambda))(\rho_{\lambda, J})$ . First,  $\text{wt } V_J(\lambda)$  is  $W_J$ -stable, hence so is  $\rho_{\lambda, J}$ . But then  $(\rho_{\lambda, J}, \alpha_i) = 0 \ \forall i \in J$ , whence  $(\rho_{\lambda, J}, -)$  is constant on  $\text{wt } V_J(\lambda)$ . Next, that the maximum value is positive for proper subsets  $\text{wt } V_J(\lambda)$  was shown in Proposition 5.1.

Now let us suppose, as in the proof of Proposition 5.1, that  $\rho_{\lambda, J} = |\text{wt } V_J(\lambda)|\lambda - \sum_{j \in J_1} m_j \alpha_j$  for positive  $m_j \in \mathbb{Z}$  and some  $J_1 \subset J$ . Thus,  $\text{wt } V_J(\lambda) = \text{wt } V_{J_1}(\lambda)$ . Now if  $i \notin J, j \in J_1$ , then  $(\alpha_j, \alpha_i) \leq 0$  (since  $J_1 \subset J$ ), so  $(\rho_{\lambda, J}, \alpha_i) \geq 0$  since  $\lambda \in P^+$ . In particular,  $\rho_{\lambda, J} \in P^+$  from above. In turn, this implies that  $\lambda \in (\text{wt } V(\lambda))(\rho_{\lambda, J})$ , and from the previous paragraph, we conclude:  $\text{wt } V_J(\lambda) \subset (\text{wt } V(\lambda))(\rho_{\lambda, J})$ .

Now suppose  $\nu \in \text{wt } V(\lambda)$  maximizes  $\rho_{\lambda, J}$ . We need to show that  $\nu \in \text{wt } V_J(\lambda)$ . We write  $\nu = \lambda - \sum_{i \in I} r_i \alpha_i$  for  $r_i \in \mathbb{Z}_+$ , and compute:

$$(\rho_{\lambda, J}, \lambda) = (\rho_{\lambda, J}, \nu) = (\rho_{\lambda, J}, \lambda) - \sum_{i \notin J_1} r_i (\rho_{\lambda, J}, \alpha_i) \leq (\rho_{\lambda, J}, \lambda),$$

since  $(\rho_{\lambda, J}, \Delta_{J_1}) = 0$  from above. Now define  $J_2 := \{i \notin J_1 : r_i > 0\}$ . The preceding equation implies that  $(\rho_{\lambda, J}, \alpha_i) = 0 \ \forall i \in J_2$ , so,

$$|\text{wt } V_J(\lambda)|(\lambda, \alpha_i) - \sum_{j \in J_1} m_j (\alpha_j, \alpha_i) = 0.$$

Since  $m_j > 0 \ \forall j \in J_1$ ,  $(\lambda, \alpha_i) = (\alpha_j, \alpha_i) = 0 \ \forall i \in J_2, j \in J_1$ .

Now let  $w_2$  be the longest element of the subgroup  $W_{J_2}$  of  $W$ . Consider  $w_2(\nu) \in \text{wt } V(\lambda)$ , where  $\nu = \lambda - \sum_{j \in J_1} r_j \alpha_j - \sum_{i \in J_2} r_i \alpha_i$ . By the previous paragraph,  $w_2(\lambda) = \lambda$  and  $w_2(\alpha_j) = \alpha_j \ \forall j \in J_1$  - and by its definition,  $w_2(\alpha_i) \in -\Delta_{J_2} \ \forall i \in J_2$ . Since  $r_i \neq 0$  for  $i \in J_2$ , this gives  $w_2(\nu) \notin \lambda - \mathbb{Z}_+ \Delta$  unless  $J_2 = \emptyset$ . Thus,  $J_2 = \emptyset$ , and  $\nu \in \text{wt } V_{J_1}(\lambda) = \text{wt } V_J(\lambda)$ . Hence  $(\text{wt } V(\lambda))(\rho_{\lambda, J}) = \text{wt } V_J(\lambda)$ , and the theorem is proved.  $\square$

**Remark.** At this point, we note that Theorem 3 does not hold for general finite-dimensional  $\mathfrak{g}$ -modules. For example, let  $\mathfrak{g}$  be of type  $A_2$ , and consider the module  $V = V(2\omega_2) \oplus V(\omega_1 + \omega_2)$ . It is easy to see that  $\{\omega_1 + \omega_2\}$  is a weak  $\mathbb{F}$ -face of  $\text{wt } V$  for all  $\mathbb{F}$ . However, the subset of  $\text{wt } V$  that maximizes the linear functional  $(\omega_1 + \omega_2, -)$  is the subset  $\{\omega_1 + \omega_2, 2\omega_2\}$ .

We now show a small result that helps classify *all* maximizer subsets inside  $\text{wt } V(\lambda)$ , for  $0 \neq \lambda \in P^+$ . Given any  $\varphi \in \mathfrak{h}_{\mathbb{R}}^*$ , the nondegeneracy of the Killing form implies that  $\varphi = (\nu, -)$ , and there exists  $w_\nu \in W$  such that  $w_\nu(\nu)$  is in the dominant Weyl chamber, i.e., in  $\mathbb{R}_+\Omega$ .

**Lemma.** *Fix  $0 \neq \lambda \in P^+$ . Then for all  $\nu \in \mathfrak{h}_{\mathbb{R}}^*$ ,*

$$(\text{wt } V(\lambda))(\nu) = w_\nu^{-1}(\text{wt } V_{I \setminus \text{supp}(w_\nu(\nu))}(\lambda)),$$

*and this map from  $\mathfrak{h}_{\mathbb{R}}^*$  to the weak  $\mathbb{F}$ -faces of  $\text{wt } V(\lambda)$  is surjective:*

$$w(\text{wt } V_J(\lambda)) = (\text{wt } V(\lambda))(w(\nu)) \quad \forall \nu \in \mathbb{R}_{>0}\Omega_{I \setminus J}.$$

In particular,  $w(\text{wt } V_J(\lambda)) = (\text{wt } V(\lambda))(w(\rho_{I \setminus J})) \quad \forall w, J$ . Moreover, Theorem 4 helps determine the answer to the question: For which (dominant)  $\mu, \nu$  are the maximizer sets the same?

*Proof.* First observe that since  $(,)$  is  $W$ -invariant and  $\text{wt } V(\lambda)$  is  $W$ -stable,

$$w(\text{wt } V(\lambda)(\nu)) = (\text{wt } V(\lambda))(w(\nu)) \quad \forall w \in W, \nu \in \mathfrak{h}_{\mathbb{R}}^*.$$

Thus, it is enough to show the first claim for dominant  $\nu$  (and  $w_\nu = 1$ ). Now, if  $\nu = \sum_i a_i \omega_i$  with  $a_i \geq 0 \quad \forall i$  and  $\mu = \lambda - \sum_{j \in I} (2b_j / (\alpha_j, \alpha_j)) \alpha_j$  with  $b_j \geq 0 \quad \forall j$ , then

$$(\nu, \mu) = (\nu, \lambda) - \sum_{i,j \in I} a_i b_j \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = (\nu, \lambda) - \sum_{i \in I} a_i b_i \leq (\nu, \lambda),$$

with equality if and only if  $a_i b_i = 0 \quad \forall i$ . This precisely means that given  $\nu$ , we must have  $b_i = 0 \quad \forall i \in \text{supp}(\nu)$ , whence we arrive at  $\text{wt } V_{I \setminus \text{supp}(\nu)}(\lambda)$ . Conversely, given that  $\text{wt } V_J(\lambda)$  is the maximizer (once again ignoring the  $w \in W$ ), we should have  $a_i = 0 \quad \forall i \in J$ , whence  $\text{supp}(\nu) = I \setminus J$ .  $\square$

**5.2.** It remains to show the last result. Once again, we need some preliminaries before proving it. Recall the definition of  $\rho_Y$  from Theorem 3.

**Proposition.** *Suppose  $0 \neq \lambda \in P^+$ .*

- (1) *Suppose  $Y$  is a  $W_J$ -stable subset of  $\text{wt } V_J(\lambda)$  for some fixed  $J \subset I$ . Then  $|Y| \rho_{\lambda, J} = |\text{wt } V_J(\lambda)| \rho_Y$ .*
- (2) *The only  $W_J$ -invariant vector inside the face  $\text{conv}_{\mathbb{R}}(\text{wt } V_J(\lambda))$  is  $\frac{1}{|\text{wt } V_J(\lambda)|} \rho_{\lambda, J}$ , which is the center of the face.*

*Proof.* The second part follows from the first, since if  $x \in \text{conv}_{\mathbb{R}}(\text{wt } V_J(\lambda))$  is  $W_J$ -invariant, then  $x = \sum_i a_i y_i$  for  $y_i \in \text{wt } V_J(\lambda)$  and  $a_i \in (0, 1)$  (and  $\sum_i a_i = 1$ ). However,

$$x = \frac{1}{|W_J|} \sum_{w \in W_J, i} a_i w(y_i),$$

whence  $x$  is an  $\mathbb{R}_+$ -linear combination of  $\rho_{Y_j}$  for distinct  $W_J$ -orbits  $Y_j \subset \text{wt } V_J(\lambda)$ . Let us write this as:  $x = \sum_j b_j (\frac{1}{|Y_j|} \rho_{Y_j})$ , with  $\sum_j b_j = 1$  (because the coefficients above added up to 1). Using this and the first part, we then get

$$x = \sum_j b_j \frac{1}{|\text{wt } V_J(\lambda)|} \rho_{\lambda, J} = \frac{1}{|\text{wt } V_J(\lambda)|} \rho_{\lambda, J}.$$



It remains to show the first part. First, if  $Y \subset \text{wt } V_J(\lambda)$  is  $W_J$ -stable (and nonempty), then  $\rho_Y$  is fixed by  $W_J$  since every  $w \in W_J$  permutes  $Y$ . Now, write  $\rho_Y = |Y|\lambda - \sum_{j \in J} a_j \alpha_j$ , for some  $a_j \in \mathbb{Z}_+$ . Then, since  $\rho_Y$  is  $W_J$ -invariant, we get:  $(\rho_Y, \alpha_j) = 0 \ \forall j \in J$ , which gives us a system of  $|J|$  linear equations in the  $|J|$  variables  $\{a_j/|Y|\}$  - namely,

$$\sum_{j \in J} (a_j/|Y|)(\alpha_j, \alpha_i) = (\lambda, \alpha_i) \ \forall i \in J.$$

We now claim that the coefficients of the  $a_j/|Y|$  are precisely the entries of the “symmetrized” Cartan matrix for  $\mathfrak{g}$ , in the rows and columns corresponding to  $J \subset I$ . But all principal minors of a symmetrized Cartan matrix of finite type are positive, so this matrix is nonsingular, which gives a unique (rational) solution to the above system. The uniqueness implies that if we start with  $\rho_{\lambda, J} = |\text{wt } V_J(\lambda)|\lambda - \sum_{i \in J} a'_i \alpha_i$ , we would get:  $a'_i/|\text{wt } V_J(\lambda)| = a_i/|Y| \ \forall i \in J$ . Thus,  $\lambda - (1/|\text{wt } V_J(\lambda)|)\rho_{\lambda, J} = \lambda - (1/|Y|)\rho_Y$ , and we are done. (Clearing the denominator of  $|Y|$  also enables us to include the case when  $Y$  is the empty set, and  $\rho_Y = 0$ .)  $\square$

We conclude this paper with the proof of our last main result.

*Proof of Theorem 4.* If  $\text{wt } V_{I_1}(\lambda) = \text{wt } V_{I_2}(\lambda)$ , then the half-sums of all the elements are clearly equal too:  $\rho_{\lambda, I_1} = \rho_{\lambda, I_2}$ . Conversely, if  $\rho_{\lambda, I_1} = \rho_{\lambda, I_2}$ , then by Theorem 3,

$$\text{wt } V_{I_1}(\lambda) = (\text{wt } V(\lambda))(\rho_{\lambda, I_1}) = (\text{wt } V(\lambda))(\rho_{\lambda, I_2}) = \text{wt } V_{I_2}(\lambda).$$

Next, if  $W_{I_1}(\lambda) = W_{I_2}(\lambda)$ , then, since  $W_{I_i}(\lambda) \subset \text{wt } V_{I_i}(\lambda)$  are  $W_{I_i}$ -stable (for  $i = 1, 2$ ), applying Proposition 5.2 twice gives

$$\frac{1}{|\text{wt } V_{I_1}(\lambda)|} \rho_{\lambda, I_1} = \frac{1}{|W_{I_1}(\lambda)|} \sum_{x \in W_{I_1}(\lambda)} x = \frac{1}{|W_{I_2}(\lambda)|} \sum_{x \in W_{I_2}(\lambda)} x = \frac{1}{|\text{wt } V_{I_2}(\lambda)|} \rho_{\lambda, I_2}.$$

Hence,  $\rho_{\lambda, I_2} \in \mathbb{Q}_{>0} \rho_{\lambda, I_1}$ , and their maximizer subsets in  $\text{wt } V(\lambda)$  coincide. By Theorem 3,  $\text{wt } V_{I_1}(\lambda) = \text{wt } V_{I_2}(\lambda)$ .

It remains to show the converse. Suppose that  $\text{wt } V_{I_1}(\lambda) = \text{wt } V_{I_2}(\lambda)$ . Recall that these sets of weights are precisely the weights of the modules  $\mathbf{U}(\mathfrak{g}_{I_1})v_\lambda$  and  $\mathbf{U}(\mathfrak{g}_{I_2})v_\lambda$ , respectively, where  $0 \neq v_\lambda$  is a highest weight vector of  $V(\lambda)$ .

Consider  $\text{conv}_{\mathbb{R}}(\text{wt } V_{I_j}(\lambda))$  as the weight polytope of  $\mathbf{U}(\mathfrak{g}_{I_j})v_\lambda$  for  $j = 1, 2$ . Since  $\mathfrak{g}_{I_1}$  and  $\mathfrak{g}_{I_2}$  are both semisimple, we can apply Theorem 1 to these polytopes. In particular, we see that the set of vertices of  $\text{conv}_{\mathbb{R}}(\text{wt } V_{I_j}(\lambda))$  is precisely  $W_{I_j}(\lambda)$ . Since  $\text{wt } V_{I_1}(\lambda) = \text{wt } V_{I_2}(\lambda)$ , these polytopes are equal, so they must have the same vertices; i.e.,  $W_{I_1}(\lambda) = W_{I_2}(\lambda)$ .  $\square$

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